Stokes theorem

S. Allais, M. Joseph

Exercise 1 (Integration theorems)**.** Let $(E, \langle \cdot\, , \cdot\rangle)$ be an oriented Euclidean space of dimension n. Let μ be its canonical volume form. Given $A \subset E$, $i_A : A \hookrightarrow E$ denotes the inclusion map.

1. Show Green-Ostrogradski theorem: given domain $\Omega \subset E$ with smooth boundary and a compactly supported vector field X,

$$\int_\Omega (\operatorname{div} X)\mu = \int_{\partial\Omega} i^*_{\partial\Omega}(X \lrcorner \mu).$$

2. Show Kelvin-Stokes theorem: if $n = 3$, given a 2-dimensional compact submanifold $\Sigma \subset E$ with boundaries,

$$\int_\Sigma i^*_\Sigma(\operatorname{curl} X \lrcorner \mu) = \int_{\partial\Sigma} i^*_{\partial\Sigma} \langle X\, , \cdot\rangle$$

3. Let (φ_t) be an isotopy of E of associated vector field (X_t). Given a bounded domain $\Omega \subset E$ with smooth boundaries, show that

$$\frac{\mathrm{d}}{\mathrm{d}s}\operatorname{Vol}(\varphi_s(\Omega))\Big|_{s=t} = \int_{\varphi_t(\Omega)} (\operatorname{div} X_t)\mu.$$

Exercise 2 (Hairy ball theorem)**.** We want to prove that in an even dimensional sphere, every vector field must vanish at some point. Assume that there exists a non-vanishing $X \in \mathcal{X}(\mathbb{S}^n)$.

1. Show that we can assume that X has constant norm equal to 1 (seeing the sphere inside $\mathbb{R}^{n+1}$).
2. Let $f : [0, 1] \times \mathbb{S}^n \to \mathbb{S}^n$ be the map

$$f_t(x) := f(t, x) = \cos(\pi t)x + \sin(\pi t)X(x).$$

   Show that f is a smooth map.

3. Compute $f_0^*\mu$ and $f_1^*\mu$ where μ is the usual volume form of $\mathbb{S}^n$. Deduce the theorem.

Exercise 3 (Haar measure)**.** Let G be a Lie group of dimension n. For $g \in G$, we denote by $L_g : G \to G$ the left multiplication by g.

1. Show that there exists a volume form $\omega \in \Omega^n(G)$ such that

$$(L_g)^*\omega = \omega, \quad \forall g \in G,$$

   which is unique up to a non-zero scalar. The measure induced by this volume form is called a Haar measure.

2. Give a Haar measure of S^1 and $GL_n(\mathbb{R})$.

Exercise 4 (Brouwer's theorem)**.**

1. Let U be an open subset of $\mathbb{R}^n$ which contains the closed unit ball. Prove that there is no smooth map $\varphi : U \to \mathbb{S}^{n-1}$ with $\varphi_{|\mathbb{S}^{n-1}} = id_{\mathbb{S}^{n-1}}$.
2. Let $\mathbb{B}(0, 1)$ be the closed euclidean ball of center 0 and radius 1 in $\mathbb{R}^n$. Let $f : \mathbb{B}(0, 1) \to \mathbb{B}(0, 1)$ be a smooth map. Prove that there exists $x \in \mathbb{B}(0, 1)$ such that $f(x) = x$.
   Hint : suppose that this is false, and build a map as in 1.